Riemann Sums

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

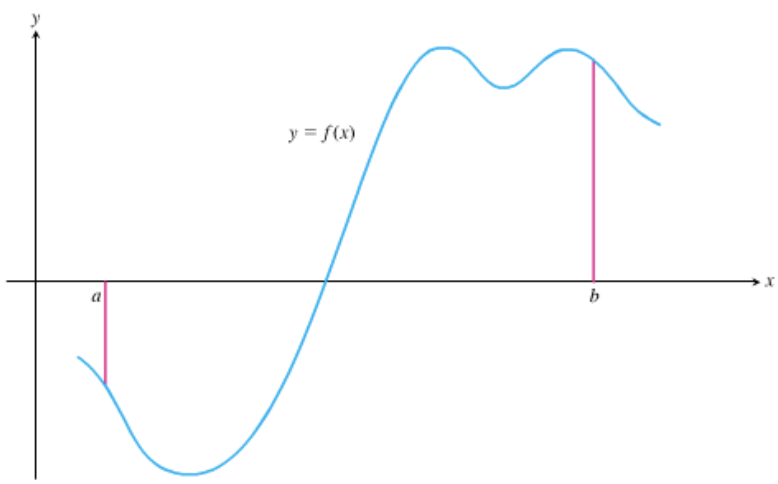


Figure 5.12 The graph of a typical function $y = f(x)$ over a closed interval $[a, b]$.

we denote a by x_0 and b by x_n

$$P = \{x_0, x_1, x_2, \cdots, x_n\}$$

a **partition** of $[a, b]$.

The partition P determines n closed **subintervals,** as shown in Figure 5.13. The k^{th} subinterval is $[x_{k-1}, x_k]$, which has length $\Delta x_k = x_k - x_{k-1}$.

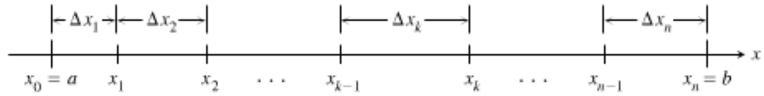


Figure 5.13 The partition $P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$ divides $[a, b]$ into n subintervals of lengths $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$. The k^{th} subinterval has length Δx_k.

In each subinterval we select some number. Denote the number chosen from the k^{th} subinterval by c_k.

Then, on each subinterval we stand a vertical rectangle that reaches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles could lie either above or below the x-axis (Figure 5.14).

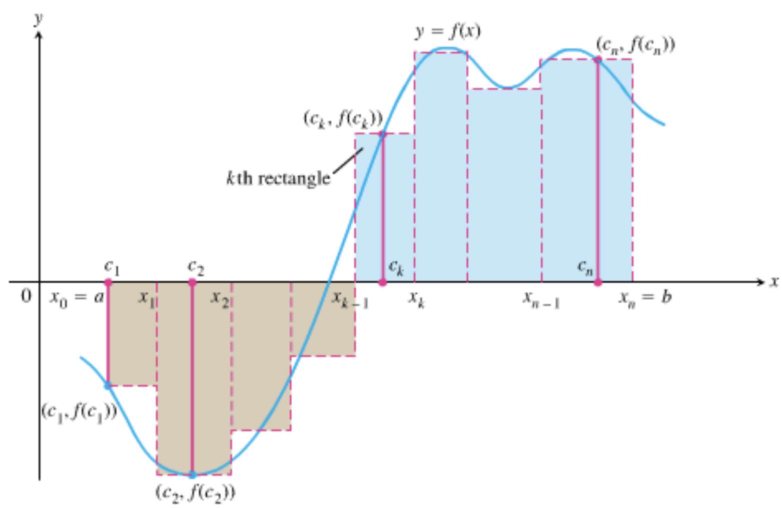


Figure 5.14 Rectangles extending from the x-axis to intersect the curve at the points $(c_k, f(c_k))$. The rectangles approximate the region between the x-axis and the graph of the function.

On each subinterval, we form the product $f(c_k) \cdot \Delta x_k$. This product can be positive, negative, or zero, depending on $f(c_k)$.

Finally, we take the sum of these products:

$$S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k.$$

This sum, which depends on the partition P and the choice of the numbers c_k, is a **Riemann sum for f on the interval $[a, b]$.**

As the partitions of $[a, b]$ become finer and finer, we would expect the rectangles defined by the partitions to approximate the region between the x-axis and the graph of f with increasing accuracy (Figure 5.15).

Just as LRAM, MRAM, and RRAM in our earlier examples converged to a common value in the limit, *all* Riemann sums for a given function on $[a, b]$ converge to a common value, as long as the lengths of the subintervals all tend to zero. This latter condition is assured by requiring the longest subinterval length (called the **norm** of the partition and denoted by $\|P\|$) to tend to zero.

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let f be a function defined on a closed interval $[a, b]$. For any partition P of $[a, b]$, let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$.

If there exists a number I such that

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

no matter how P and the c_k's are chosen, then f is **integrable** on $[a, b]$ and I is the **definite integral** of f over $[a, b]$.

THEOREM 1 The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definite integral of f over $[a, b]$ is given by

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

Upper limit of integration

The function is the **integrand.**

Integral sign

x is the **variable of integration.**

$$\underbrace{\int_a^b f(x)\,dx}_{\textbf{Integral} \text{ of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have **evaluated the integral.**

EXAMPLE 1 Using the Notation

The interval $[-1, 3]$ is partitioned into n subintervals of equal length $\Delta x = 4/n$. Let m_k denote the midpoint of the k^{th} subinterval. Express the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} (3(m_k)^2 - 2m_k + 5)\,\Delta x$$

as an integral.

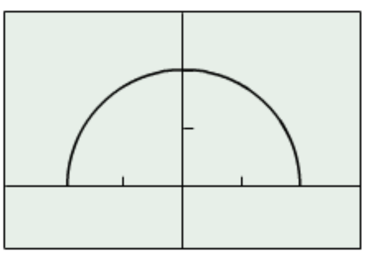
DEFINITION Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ from a to b** is the integral of f from a to b,

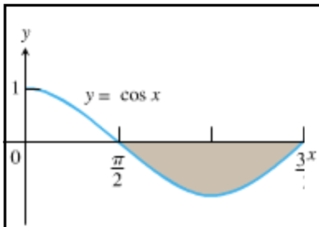
$$A = \int_a^b f(x)\,dx.$$

EXAMPLE 2 Revisiting Area Under a Curve

Evaluate the integral $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$.



[−3, 3] by [−1, 3]



gure 5.18 Because $f(x) = \cos x$ npositive on $[\pi/2, 3\pi/2]$, the inte of a negative number. The area of th ded region is the opposite of this egral,

$$\text{Area} = -\int_{\pi/2}^{3\pi/2} \cos x \, dx.$$

If an integrable function $y = f(x)$ is nonpositive, the nonzero terms $f(c_k)\Delta x_k$ in the Riemann sums for f over an interval $[a, b]$ are negatives of rectangle areas. The limit of the sums, the integral of f from a to b, is therefore the *negative* of the area of the region between the graph of f and the x-axis (Figure 5.18).

$$\int_a^b f(x)\,dx = -(\text{the area}) \quad \text{if} \quad f(x) \le 0.$$

Or, turning this around,

$$\text{Area} = -\int_a^b f(x)\,dx \quad \text{when} \quad f(x) \le 0.$$

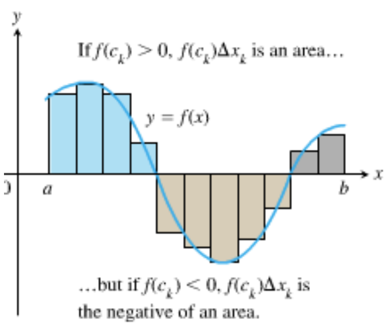


Figure 5.19 An integrable function f with negative as well as positive values.

If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a, b]$, then the Riemann sums for f on $[a, b]$ add areas of rectangles that lie above the x-axis to the negatives of areas of rectangles that lie below the x-axis, as in Figure 5.19. The resulting cancellations mean that the limiting value is a number whose magnitude is less than the total area between the curve and the x-axis. The value of the integral is the area above the x-axis minus the area below.

For any integrable function,

$$\int_a^b f(x)\,dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$

THEOREM 2 The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a).$$

EXAMPLE 3 Revisiting the Train Problem

A train moves along a track at a steady 75 miles per hour from 7:00 A.M. to 9:00 A.M. Express its total distance traveled as an integral. Evaluate the integral using Theorem 2.

$v(t) = 75$

$$\int_7^9 75\,dt = 75(9-7)$$

$$= \boxed{150 \text{ miles}}$$

Evaluate the following integrals numerically.

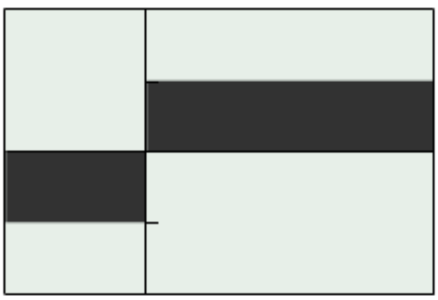
(a) $\int_{-1}^{2} x \sin x \, dx$ **(b)** $\int_{0}^{1} \frac{4}{1+x^2} \, dx$ **(c)** $\int_{0}^{5} e^{-x^2} \, dx$

≈ 2.04 ≈ 3.14 ≈ 0.89

EXAMPLE 5 **Integrating a Discontinuous Function**

Find $\int_{-1}^{2} \frac{|x|}{x}\,dx.$

$y = |x|/x$



[−1, 2] by [−2, 2]

Figure 5.23 A discontinuous integrable function:

$$\int_{-1}^{2} \frac{|x|}{x}\,dx = -(\text{area below } x\text{-axis}) + (\text{area above } x\text{-axis}).$$

(Example 5)